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Research Article

ON THE TERNARY QUADRATIC DIOPHANTINE EQUATION $2(x^2 + y^2) - 3xy = 11z^2$

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ABSTRACT

The ternary quadratic Diophantine equation $2(x^2 + y^2) - 3xy = 11z^2$ is analyzed for its patterns of nonzero distinct integral solutions. A few interesting properties among the solutions and polygonal numbers are presented.

Keywords: Ternary quadratic, integral solutions.

MSC CODE:

INTRODUCTION

A Diophantine Equation is an equation relating integer (or sometimes Natural number or Whole number) quantities. Finding the solution or solutions to a Diophantine equation is closely tied to modular arithmetic and number theory. Often when a Diophantine equation has infinitely many solutions, parametric form is used to express the relation between the variables of the equation. Diophantine Equation are named for the ancient Greek/ Alexandrian Mathematicians Diaphanous.

The Ternary Quadratic Diophantine Equation offers an unlimited field for research because of their variety [1, 2]. For an extensive review of various problems, one may refer [2-10].

This communication concerns with yet another interesting Ternary Quadratic Equation $2(x^2+y^2)-3xy=11z^2$ for determining its infinitely many non-zero integral solutions. Also, a few interesting relations among the solutions are presented.

1. METHOD OF ANALYSIS

The ternary quadratic equation to be solved for its integer solution is

$$2(x^2 + y^2) - 3xy = 11z^2 \tag{1}$$

The substitution of linear transformations,

$$x = u + v;\ y = u - v \tag{2}$$

in (1) leads to

$$u^2 + 7v^2 = 11z^2 \tag{3}$$

METHOD: 1.1

Assume that,

$$z = a^2 + 7b^2;\ \text{a, b} \neq 0 \tag{4}$$

And write 11 in (1) as,

$$11 = (2 + i\sqrt{7})\,(2 - i\sqrt{7}) \tag{5}$$

Using (4) & (5) in (3) and applying the method of factorization,

$$u^2 + 7v^2 = (a^2 + 7b^2)^2\,(2 + i\sqrt{7})\,(2 - i\sqrt{7}) \tag{6}$$

Define from (6),

$$u + i\sqrt{7}v = (2 + i\sqrt{7})\,(a + i\sqrt{7}b)^2$$

$$(u - i\sqrt{7}v) = (2 - i\sqrt{7})(\,a - i\sqrt{7}b)^2$$

Equating the real and imaginary parts, we have

$$u = u(a,b) = 2a^2 - 14b^2 - 14ab$$

$$v = v(a,b) = a^2 - 7b^2 + 4ab$$

Then using (2), the values of x and y are given by,

$$x = x(a,b) = 3a^2 - 21b^2 - 10ab$$

$$y = y(a,b) = a^2 - 7b^2 - 18ab$$

As our aim is to find integer solutions, choose a and b so that x and y are integers.

Properties: 1.1

a. $x(a,1) - t_{4,a} - t_{6,a} \equiv 0 \pmod 7$

b. $z(a,a+1) - t_{10,a} - t_{10,a} \equiv 0 \pmod 7$

c. $y(1,b) + t_{8,b} - t_{10,b} \equiv 1 \pmod 2$

d. $z(1,b) - x(1,b) - t_{16,a} - t_{44,a} \equiv 0 \pmod 2$

METHOD: 1.2

Write (3) as

$$u^2 - (2z)^2 = 7(z^2 - v^2)$$

Factorizing the above, it is written $\frac{u-2z}{z-v} = \frac{7(z+v)}{u+2z} = \frac{\alpha}{\beta}$ as, $\beta \neq 0$

which is equivalent to the following two equations,

$$-\alpha u + 7\beta v + (7\beta - 2\alpha)z = 0 \quad (7)$$

$$\beta u + \alpha v - (2\beta + 2\alpha)z = 0 \quad (8)$$

By applying the method of cross multiplication, we get the integral solutions of (1) to be

$$x = 3\alpha^2 - 21\beta^2 - 10\alpha\beta$$

$$y = \alpha^2 - 7\beta^2 - 18\alpha\beta$$

$$z = -\alpha^2 - 7\beta^2$$

Properties: 1.2

a. $x(1,\beta) + t_{32,\beta} + t_{14,\beta} \equiv 0 \pmod 3$

b. $x(\alpha,1) + z(\alpha,1) - t_{4,\alpha} - t_{4,\alpha} \equiv 0 \pmod 2$

c. $z(\alpha,\alpha+1) + t_{4,\alpha} + t_{16,\alpha} \equiv 0 \pmod 7$

d. $y(1,\beta) + t_{8,\beta} + t_{10,\beta} \equiv 1 \pmod 3$

METHOD: 1.3

Write (3), in the form of ratio as

$\frac{(u+2z)}{(z-v)} = \frac{7(z+v)}{(u-2z)} = \frac{\alpha}{\beta}$, $\beta \neq 0$ which is equivalent to the following two equations,

$$-\alpha u + 7\beta v + (2\alpha + 7\beta)z = 0 \quad (9)$$

$$\beta u + \alpha v + (2\beta - \alpha) z = 0 \quad (10)$$

By applying the method of cross multiplication, we get the integral solutions of (1) to be

$$x = x(\alpha,\beta) = -\alpha^2 + 7\beta^2 - 18\alpha\beta$$

$$y = y(\alpha,\beta) = -3\alpha^2 + 21\beta^2 - 10\alpha\beta$$

$$z = z(\alpha,\beta) = -\alpha^2 - 7\beta^2$$

Properties: 1.3

a. $x(1,\beta) - t_{8,\beta} - t_{10,\beta} \equiv 1 \pmod 3$

b. $y(\alpha,1) + t_{4,\alpha} + t_{6,\alpha} \equiv 0 \pmod 7$

c. $z(\beta+1,\beta) + t_{4,\beta} + t_{16,\beta} \equiv 1 \pmod 2$

d. $x(1,\beta) + y(1,\beta) - t_{18,\beta} - t_{42,\beta} \equiv 0 \pmod 4$

CONCLUSION

We conclude that, there is no general rule on which method to use and the best way to gain experience is by practice. Some of the methods to solve ternary quadratic Diophantine equation is illustrated. One may search for other methods of solutions and their corresponding properties.

REFERENCES

[1]L.E. Dickson, "*History of Theory of Numbers*", (Vol.2, Chelsea Publishing Company, New York, 1952).

[2] L.J. Mordell, "*Diophantine Equations*", (Academic Press, London, 1969).

[3]P.Bachmann, "*Das Fermat problem in Seiner Bisherigen Entwicklung*", (Walter de Gruyter & Co., Berlin and Leipzing, 1966).

[4]L.E Dickson, "*Modern Elementary Theory of Numbers*", (Uni. Chicago Press, Chicago, 1939).

[5]A.O.Gelfond, "*The solution of Equations in Integer*", (Freeman & Co., San Francisco and London, 1961).

[6]G.H.Hardy and E.M. Wright, "*The Theory of Numbers*", (Clarendon Press, Oxford, 1945).

[7]L.J.Mordell, "*Chapter in the Theory of Numbers*", (Cambridge University Press, Cambridge, 1947).

[8]T.Nagell, "*Introduction to Number Theory*", (Almqvist and Wiksell, Stockholm; John Wiley and Sons Inc., New York, 1959).

[9]T. Skolem, "*Diophantische Gleichungen*", (Springer, Berlin, 1938).

[10]J.V. Uspensky and M.A. Heaslet, "*Elementary Number Theory*", (McGraw Hill Book Co. Inc., New York, 1939).