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Research Article

NUMERICAL SOLUTION OF FUZZY DIFFERENTIAL EQUATION BY

RUNGE-KUTTA RALSTON METHOD

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ABSTRACT

This paper attempts to propose a new method of computing approximation of the solution for fuzzy differential equation with initial conditions using Runge-Kutta Ralston in order to increase the order of the accuracy of the solution. This method is discussed in detail followed by a complete error analysis.

Keywords

Fuzzy Differential Equations, Fuzzy Cauchy problem, Runge-Kutta Ralston Method, Error Analysis

1. Preliminaries

Consider the initial value problem

$$y'(t) = \begin{cases} f(t, y(t)), & t_0 \le t \le b \\ y(t_0) = y_0 \end{cases} \qquad (2.1)$$

We assume that

1. $f(t, y(t))$ is defined, continuous in the strip with $t_0 \le t \le b$, $-\infty < y < \infty$ with t_0 and b are finite.
2. There exists a constant L such that for any t in $[t_0, b]$ and any two numbers y and y^*
$\left| f(t,y) - f(t,y^*) \right| \le L \left| y - y^* \right|$

These conditions are sufficient to prove that $\exists$ on $[t_0, b]$ a unique continuous differentiable function y(t) satisfying (2.1)

The basis of all Runge-Kutta methods is to express the difference between the value of y at t_{n+1} and t_n as $y_{n+1} - y_n = \sum_{i=0}^{m} w_i k_i$; where w_i are constant and $k_i = hf(t_n + a_i h, y_n + \sum_{j=1}^{i-1} c_{ij} k_j)$

Most efforts to increase the order of accuracy of the Runge-Kutta methods have been accomplished by increasing the number of Taylor's series terms used and thus the number of functional evaluations required [5]. The method proposed in [16] introduces new terms involving higher order derivatives of 'f' in the Runge-Kutta k_i terms (i > 1) to obtain a higher order of accuracy without a corresponding increase in evaluations of f, but with the addition of evaluations of 'f' by Runge-Kutta Ralston method for autonomous system proposed in [16].

Consider $y(t_{n+1}) = y(t_n) + w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4$ where

$k_1 = hf(t_n, y(t_n))$

$k_2 = hf(t_n+c_2h, y(t_n)+a_{21}k_1)$

$k_3 = hf(t_n+c_3h, y(t_n)+a_{31}k_1+a_{32}k_2)$

$k_4 = hf(t_n+c_4h, y(t_n)+a_{41}k_1+a_{42}k_2+a_{43}k_3)$

Utilizing the Taylor's series expansion techniques, Runge-Kutta Ralston method is given by,

$y_{n+1} = y_n + 0.17476028\, k_1 - 0.55148066\, k_2 + 1.20553560\, k_3 + 0.17118478\, k_4$

$k_1 = hf(t_n, y(t_n))$

$k_2 = hf(t_n + 0.4h, y(t_n) + 0.4\, k_1)$

$k_3 = hf(t_n + 0.45573725h, y(t_n) + 0.29697761\, k_1 + 0.15875964\, k_2)$

$k_4 = hf(t_n + h, y(t_n) + 0.21810040\, k_1 - 3.05096516\, k_2 + 3.83286476\, k_3)$

Definition – 1.1

A fuzzy number u as a fuzzy subset of R ie) u : R → [0, 1] satisfying the following conditions.

i). u is normal, ie $\exists\ x_0 \in R \ni u(x_0) = 1$
ii). u is a convex fuzzy set ie) $u(tx + (1-t)y) \geq \min\{u(x), u(y)\}$, $\forall\ t \in [0, 1]$ and $x, y \in R$
iii). u is upper semi continuous on R
iv). $\overline{\{x \in R, u(x) > 0\}}$ is compact

The set E is the family of fuzzy numbers and arbitrary fuzzy number is represented by an ordered pair of functions $\left(\underline{u}(r), \overline{u}(r)\right)$, $0 \leq r \leq 1$ has satisfies the following requirements

1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over [0, 1] w.r.to any 'r'.
2. $\overline{u}(r)$ is a bounded right continuous non-increasing function over [0, 1] w.r.to any 'r'.
3. $\underline{u}(r) \leq \overline{u}(r)$, $0 \leq r \leq 1$, r-level cut is $[u]_r = \{x/u(x) \geq r\}$, $0 \leq r \leq 1$ as a closed & bounded interval denoted by $[u]_r = \left[\underline{u}(r), \overline{u}(r)\right]$ and $[u]_0 = \{x/u(x) > 0\}$ is compact.

Definition – 1.2

A triangular fuzzy number u is a fuzzy set in E that is characterised by an ordered triple $(u_l, u_c, u_r) \in R^3$ with $u_l < u_c < u_r$ such that $[u]_0 = [u_l : u_r]$ and $[u]_1 = [u_c]$. The membership function of the triangular fuzzy number u is given by

$$u(x) = \begin{cases} \dfrac{x - u_l}{u_c - u_l}, & u_l \leq x \leq u_c \\ 1 & x = u_c \\ \dfrac{u_r - x}{u_r - u_c}, & u_c \leq x \leq u_r \end{cases}$$ and we will have

$u > 0$ if $u_l > 0$, $u \geq 0$ if $u_l \geq 0$, $u < 0$ if $u_c < 0$ and $u \leq 0$ if $u_c \leq 0$

Let I be a real interval. A mapping y : I → E as called a fuzzy process and its α - level set is denoted by $[y(t)]_\alpha = \left[\underline{y}(t, y), \overline{y}(t, y)\right]$, $t \in I$, $0 < \alpha \leq I$. The Seikkala derivative y(t) of a fuzzy process is defined by $\left[y^1(t)\right]_\alpha = \left[\underline{y}^1(t, y), \overline{y}^1(t, y)\right]$, $t \in I$, $0 < \alpha \leq I$ provided the equation defines fuzzy number as in [12]. For $u, v \in E$ and $\lambda \in \Re$, the $u + v$ and the product λu can be defined by $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda[u]_\alpha$ where $\alpha \in [0, 1]$, $[u]_\alpha + [v]_\alpha$ means the addition of two intervals of $\Re$ and $[u]_\alpha$ means the product between a scalar and a subset of $\Re$. Arithmetic operation of arbitrary fuzzy numbers

u = $\left(\underline{u}(r), \overline{u}(r)\right)$ and v = $\left(\underline{v}(r), \overline{v}(r)\right)$ and $\lambda \in \Re$ can be defined as

i). u = v if $\underline{u}(r) = \underline{v}(r)$ and $\overline{u}(r) = \overline{v}(r)$
ii). u + v = $\left(\underline{u}(r) + \underline{v}(r), \overline{u}(r) + \overline{v}(r)\right)$
iii). u - v = $\left(\underline{u}(r) - \overline{v}(r), \overline{u}(r) - \underline{v}(r)\right)$
iv). λu = $\left(\lambda \underline{u}(r), \lambda \overline{u}(r)\right)$ if $\lambda \geq 0$
= $\left(\lambda \overline{u}(r), \lambda \underline{u}(r)\right)$ if $\lambda < 0$

2. A Fuzzy Cauchy Problem

Consider the fuzzy initial value problem

$$y'(t) = \begin{cases} f(t, y(t)), \; 0 \leq t \leq T \\ y(0) = y_0 \end{cases} \qquad (3.1)$$

where f is a continuous mapping from R x R → R and $y_0 \in E$ with r-level sets $[y_0]_r = \left[\underline{y}(0:r), \overline{y}(0:r)\right]$, $r \in [0, 1]$. The extension principle of Zadeh leads to the definition of f(t, y) then y = y(t) is a fuzzy number.

$$f(t, y)(s) = \sup\{y(\tau) \backslash s = f(t, \tau)\}, \; s \in R$$

$$\left[f(t,y)\right]_r = \left[\underline{f}(t, y:r), \overline{f}(t, y:r)\right], \; r \in [0, 1]$$

It follows that

$$\underline{f}(t, y:r) = \min\left\{f(t,u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$$

$$\overline{f}(t, y:r) = \max\left\{f(t,u) \backslash u \in \left[\underline{y}(r), \overline{y}(r)\right]\right\}$$

Theorem:

Let f satisfy $\left|f(t,v) - f(t,\overline{v})\right| \leq g\left(t, \left|v - \overline{v}\right|\right)$, $t \geq 0$ and v, $\overline{v} \in R$, where g : $R_+ \times R_+ \to R_+$ is a continuous mapping such that r → g(t, r) is non-decreasing an initial value problem has a solution on R_+ or $u_0 > 0$ and that $u(t) \equiv 0$ is the only solution of (3.3) for $u_0 = 0$ then the fuzzy initial value problem (3.1) has a unique fuzzy solution.

$u^1(t) = g(t, u(t)), \; u(0) = u_0$

3. Runge–Kutta Ralston Method

Let the exact solution of the given equation $[y(t)]_r = \left[\underline{y}(t:r), \overline{y}(t:r)\right]$ is approximated by some $[y(t)]_r = \left[\underline{y}(t:r), \overline{y}(t:r)\right]$ and we define

$$\underline{y}(t_{n+1}:r) - \underline{y}(t_n:r) = \sum_{i=1}^{4} w_i \underline{k_i}$$

$$\overline{y}(t_{n+1}:r) - \overline{y}(t_n:r) = \sum_{i=1}^{4} w_i \overline{k_i}$$

Where w_i's are constant and

$$\left[k_i(t, y(t,r))\right]_r = \left[\underline{k_i}(t, y(t,r)), \overline{k_i}(t, y(t,r))\right] \text{where i = 1, 2, 3 and 4}$$

$$\underline{k_1}(t, y(t:r)) = hf\left(t_n, \underline{y}(t_n:r)\right)$$

$$\overline{k_1}(t, y(t:r)) = hf\left(t_n, \overline{y}(t_n:r)\right)$$

$$\underline{k_2}(t, y(t:r)) = hf\left(t_n + 0.4h, \underline{y}(t_n:r) + 0.4\underline{k_1}\right)$$

$$\overline{k_2}(t, y(t:r)) = hf\left(t_n + 0.4h, \overline{y}(t_n:r) + 0.4\overline{k_1}\right)$$

$$\underline{k_3}(t, y(t:r)) = hf\left(t_n + 0.45573725h, \underline{y}(t_n:r) + 0.29697761\underline{k_1} + 0.15875964\underline{k_2}\right)$$

$$\overline{k_3}(t, y(t:r)) = hf\left(t_n + 0.45573725h, \overline{y}(t_n:r) + 0.29697761\overline{k_1} + 0.15875964\overline{k_2}\right)$$

$$\underline{k_4}(t, y(t:r)) = hf\left(t_n + h, \underline{y}(t_n:r) + 0.21810040\underline{k_1} - 3.05096516\underline{k_2} + 3.83286476\underline{k_3}\right)$$

$$\overline{k_4}(t, y(t:r)) = hf\left(t_n + h, \overline{y}(t_n:r) + 0.21810040\overline{k_1} - 3.05096516\overline{k_2} + 3.83286476\overline{k_3}\right)$$

$$F(t,y(t:r)) = 0.17476028\ \underline{k_1} - 0.55148066\ \underline{k_2} + 1.20553560\ \underline{k_3} + 0.17118478\ \underline{k_4}$$

$$G(t,y(t:r)) = 0.17476028\ \overline{k_1} - 0.55148066\ \overline{k_2} + 1.20553560\ \overline{k_3} + 0.17118478\ \overline{k_4}$$

The exact and approximate solution at t_n, $0 \le n \le N$ are denoted by

$[Y(t_n)]_r = \left[\underline{Y}(t_n : r), \overline{Y}(t_n : r)\right]$ and $[y(t_n)]_r = \left[\underline{y}(t_n : r), \overline{y}(t_n : r)\right]$ respectively.

The solution calculated by grid points at $a = t_0 \le t_1 \le t_2 \le . . . \le t_N = b$ and $h = \dfrac{b-a}{N}$ $= t_{n+1} - t_n$. Therefore we have

$$\underline{y}(t_{n+1} : r) = \underline{y}(t_n : r) + F\left[t_n, \underline{y}(t_n : r)\right]$$

$$\overline{y}(t_{n+1} : r) = \overline{y}(t_n : r) + G\left[t_n, \overline{y}(t_n : r)\right]$$

To show the convergence of these approximation

ie) $\lim_{h\to 0} \underline{y}(t : r) = \underline{Y}(t : r)$ and

$\lim_{h\to 0} \overline{y}(t : r) = \overline{Y}(t : r)$

Lemma:

Let a sequence of numbers $\{W\}_{n=0}^{N}$ satisfy $|W_{n+1}| \le A|W_n| + B$, $0 \le n \le N-1$ or some given positive constants A and B then $|W_n| \le A^n |W_0| + B\dfrac{A^n - 1}{A-1}$, $0 \le n \le N-1$ [13]

Lemma:

Let a sequence of numbers $\{W\}_{n=0}^{N}$ and $\{V\}_{n=0}^{N}$ satisfy the condition $|W_{n+1}| \le |W_n| + A\max\{|W_n|, |V_n|\} + B$ and $|V_{n+1}| \le |V_n| + A \max\{|W_n|, |V_n|\} + B$ for some given positive constants A and B and denote $U_n = |W_n| + |V_n|$, $0 \le n \le N$ where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$

Theorem:

Let F(t, u, v) and G(t, u, v) belongs to $C^4(K)$ and let the partial derivatives of F and G be bounded over K, then for arbitrary fixed value r, $0 \le r \le 1$ are approximate solutions converge to the exact solutions of $\underline{Y}(t_n : r)$ and $\overline{Y}(t_n : r)$ uniformly in t.

Numerical Example

Consider $y^1(t) = y(t)$, $t \in [0, 1]$ with $y(0) = (0.75 + 0.25r, 1.2 - 0.20r)$ where $0 \le r \le 1$

Solution: The exact solution is given by $\underline{y}(t:r) = \underline{y}(t:r)e^t$ & $\overline{y}(t:r) = \overline{y}(t:r)e^t$, at t = 1, y(1:r)=[(0.75+0.25r)e,(1.2-0.2r)e], 0≤r≤1. The exact & approximate solution obtained by the Runge-Kutta Ralston method and Euler method with h=0.1 are given below

Table – 5.1

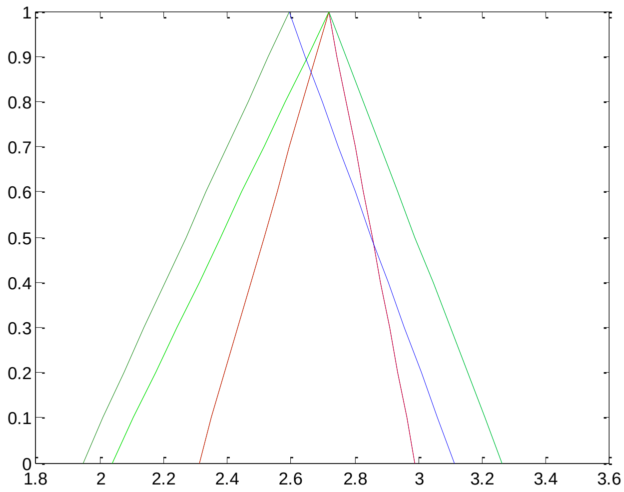
r	Exact Solution		Runge-Kutta Ralston		Euler Method	
	$\underline{Y}$	$\overline{Y}$	$\underline{Y}$	$\overline{Y}$	$\underline{Y}$	$\overline{Y}$
0.0	2.310539554	2.990110011	2.0387098789	3.2619359493	1.945306659	3.112491131
0.1	2.351313782	2.962927193	2.1066668034	3.2075695992	2.010150433	3.060616016
0.2	2.392088009	2.935744375	2.1746239662	3.1532044411	2.074994087	3.008740902
0.3	2.432862237	2.908561557	2.2425806522	3.0988388062	2.139837265	2.956866026
0.4	2.473636464	2.881378738	2.3105380535	3.0444731712	2.204681158	2.904991627
0.5	2.514410691	2.85419592	2.3784949780	2.9901077747	2.269524097	2.853116989
0.6	2.555184919	2.827013102	2.4464519024	2.9357423782	2.334367991	2.801241875
0.7	2.595959146	2.799830283	2.5144088268	2.8813762665	2.399211884	2.74936676
0.8	2.636733374	2.772647465	2.5823657513	2.8270103931	2.464055538	2.697492123
0.9	2.677507601	2.745464647	2.6503226757	2.7726454735	2.528898954	2.645617485
1.0	2.718281829	2.718281829	2.7182798386	2.7182798386	2.593742609	2.593742609

The error between the approximate solution by Euler method and the exact solution and also the error between the approximate solution by Runge-Kutta Ralston method and the exact solution of the problem are listed below.

Table – 5.2

r	Exact Solution		Runge-Kutta Ralston	
	$\underline{Y}$	$\overline{Y}$	$\underline{Y}$	$\overline{Y}$
0.0	0.2718296751	0.2718259383	0.365232895	0.122381120
0.1	0.2446469786	0.2446424062	0.341163349	0.097688823
0.2	0.2174640428	0.2174600661	0.317093922	0.072996527
0.3	0.1902815848	0.1902772492	0.293024972	0.048304469
0.4	0.1630984105	0.1630944332	0.268955306	0.023612889
0.5	0.1359157130	0.1359118547	0.244886594	0.001078931
0.6	0.1087330166	0.1087292762	0.220816928	0.025771227
0.7	0.0815503192	0.0815459835	0.196747262	0.050463523
0.8	0.0543676227	0.0543629281	0.172677836	0.075155342
0.9	0.0271849253	0.0271808265	0.148608647	0.099847162
1.0	0.0000019904	0.0000019904	0.124539220	0.124539220

Figure – 5.1



5. CONCLUSION

In this work, we have used the proposed fifth order Runge-Kutta Ralston method to find the numerical solutions of fuzzy differential equations. Taking into account the convergence order of the Euler method is O(h), a higher order of convergence O(h^3) is obtained by the proposed method and by the method proposed in [15]. Comparison of the solutions of example shows that the proposed method gives a better solution than the Euler method.

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